1

$$(1+x)^{-2}$$

$$1+(-2)x+\frac{(-2)(-3)}{2}x^2+\frac{(-2)(-3)(-4)}{6}x^3$$

$$1-2x+3x^2-4x^3, \quad |x|<1$$

(Total for question 1 is 3 marks)

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2 (a) $4^{-\frac{1}{2}}(1+\frac{3}{2}x)^{-\frac{1}{2}}$

$$\frac{1}{2}\left(1+\left(-\frac{1}{2}\right)\left(\frac{3}{2}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(\frac{3}{2}x\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\left(\frac{3}{2}x\right)^3\right)$$

$$\frac{1}{2}\left(1-\frac{3}{4}x+\frac{27}{8}x^2-\frac{135}{128}x^3\right)$$

$$\frac{1}{2}-\frac{3}{8}x+\frac{27}{16}x^2-\frac{135}{256}x^3$$

(b) $|\frac{3}{2}x|<1$

$$|x|<\frac{2}{3}$$

(Total for question 2 is 5 marks)

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3 (a) $8^{\frac{1}{3}}(1+\frac{3}{2}x)^{\frac{1}{3}}$

$$2\left(1+\left(\frac{1}{3}\right)\left(\frac{3}{2}x\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}\left(\frac{3}{2}x\right)^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}\left(\frac{3}{2}x\right)^3\right)$$

$$2\left(1+\frac{1}{2}x-\frac{1}{4}x^2+\frac{5}{24}x^3\right)$$

$$2+x-\frac{1}{2}x^2+\frac{5}{12}x^3$$

(b) $|\frac{3}{2}x|<1$

$$|x|<\frac{2}{3}$$

(Total for question 3 is 5 marks)

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4 (a) $(1+2x)^{\frac{1}{3}}$

$$1+\left(\frac{1}{3}\right)(2x)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}(2x)^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}(2x)^3$$

$$1+\frac{2}{3}x-\frac{4}{9}x^2+\frac{40}{81}x^3 \quad |x|<\frac{1}{2}$$

(b) $1+2x=1.1 \implies x=0.05$

$$1+\frac{2}{3}(0.05)-\frac{4}{9}(0.05)^2+\frac{40}{81}(0.05)^3$$

$$1.03228$$

(Total for question 4 is 7 marks)

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5 (a) $(2+3x)^{-2}=2^{-2}\left(1+\frac{3}{2}x\right)^{-2}$

$$\frac{1}{4}\left(1+(-2)\left(\frac{3}{2}x\right)+\frac{(-2)(-3)}{2}\left(\frac{3}{2}x\right)^2\right)$$

$$\frac{1}{4}\left(1-3x+\frac{27}{4}x^2\right)$$

$$\frac{1}{4}-\frac{3}{4}x+\frac{27}{16}x^2$$

(b) $\left|\frac{3}{2}x\right|<1 \implies |x|<\frac{2}{3}$

(Total for question 5 is 5 marks)

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6 (a) $3^{\frac{1}{2}}\left(1+\frac{2}{3}x\right)^{\frac{1}{2}}$

$$\sqrt{3}\left(1+\frac{1}{2}\left(\frac{2}{3}x\right)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2}\left(\frac{2}{3}x\right)^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}\left(\frac{2}{3}x\right)^3\right)$$

$$\sqrt{3}\left(1+\frac{1}{3}x-\frac{1}{18}x^2+\frac{1}{54}x^3\right)$$

$$\sqrt{3}+\frac{\sqrt{3}}{3}x-\frac{\sqrt{3}}{18}x^2+\frac{\sqrt{3}}{54}x^3$$

(b) $\left|\frac{2}{3}x\right|<1 \implies |x|<\frac{3}{2}$

(c) When $x=0.1$, $\sqrt{3+2(0.1)}=\sqrt{3.2}=\sqrt{\frac{16}{5}}=\frac{4}{\sqrt{5}}=\frac{4\sqrt{5}}{5}$

$$\frac{4\sqrt{5}}{5}=\sqrt{3}+\frac{\sqrt{3}}{3}(0.1)-\frac{\sqrt{3}}{18}(0.1)^2+\frac{\sqrt{3}}{54}(0.1)^3$$

$$\frac{4}{5}\sqrt{5}\approx 1.788855... \implies \sqrt{5}\approx 2.236$$

(Total for question 6 is 8 marks)

7

(a) $1 + (-3)(3x) + \frac{(-3)(-4)}{2}(3x)^2 + \frac{(-3)(-4)(-5)}{6}(3x)^3$

$$1 - 9x + 54x^2 - 270x^3$$

(b) $|3x| < 1 \implies |x| < \frac{1}{3}$

(c)

$$(2x + 1)(1 - 9x + 54x^2 - 270x^3)$$

$$2x - 18x^2 + 108x^3 + 1 - 9x + 54x^2 - 270x^3$$

$$1 - 7x + 36x^2 - 162x^3$$

(Total for question 7 is 8 marks)

8

(a) $9^{\frac{1}{2}}(1 - \frac{2}{9}x)^{\frac{1}{2}} = 3(1 + (\frac{1}{2})(-\frac{2}{9}x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-\frac{2}{9}x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}(-\frac{2}{9}x)^3)$

$$3(1 - \frac{1}{9}x - \frac{1}{162}x^2 - \frac{1}{1458}x^3) = 3 - \frac{1}{3}x - \frac{1}{54}x^2 - \frac{1}{486}x^3$$

(b) $|\frac{2}{9}x| < 1 \implies |x| < 4.5$

(c) $9 - 2x = 8.9 \implies 2x = 0.1 \implies x = 0.05$

$$3 - \frac{1}{3}(0.05) - \frac{1}{54}(0.05)^2 - \frac{1}{486}(0.05)^3 \approx 2.9833$$

(Total for question 8 is 7 marks)

9

(a) $(1 + 5x)^{\frac{1}{2}}(1 - x)^{-\frac{1}{2}}$

$$(1 + 5x)^{\frac{1}{2}} = 1 + \frac{5}{2}x - \frac{25}{8}x^2$$

$$(1 - x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

Product: $(1 + \frac{5}{2}x - \frac{25}{8}x^2)(1 + \frac{1}{2}x + \frac{3}{8}x^2) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{2}x + \frac{5}{4}x^2 - \frac{25}{8}x^2 = 1 + 3x - \frac{3}{2}x^2$

(b) The expansion of $(1 + 5x)^{\frac{1}{2}}$ is only valid for $|5x| < 1$, i.e., $|x| < \frac{1}{5}$. Since $x = \frac{1}{2} > \frac{1}{5}$, the expansion does not converge.

(c) When $x = \frac{1}{9}$, $\sqrt{\frac{1+5/9}{1-1/9}} = \sqrt{\frac{14/9}{8/9}} = \sqrt{\frac{14}{8}} = \sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{2}$

$$\frac{\sqrt{7}}{2} = 1 + 3(\frac{1}{9}) - \frac{3}{2}(\frac{1}{9})^2 = 1 + \frac{1}{3} - \frac{1}{54} = \frac{54 + 18 - 1}{54} = \frac{71}{54}$$

$$\sqrt{7} = \frac{71}{27}$$

(Total for question 9 is 10 marks)

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10 (a)

$$(9+x)^{-\frac{1}{2}} = 9^{-\frac{1}{2}}(1+\frac{1}{9}x)^{-\frac{1}{2}}$$

$$\frac{1}{3}(1+(-\frac{1}{2})(\frac{1}{9}x)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\frac{1}{9}x)^2)$$

$$\frac{1}{3}(1-\frac{1}{18}x+\frac{1}{216}x^2) = \frac{1}{3}-\frac{1}{54}x+\frac{1}{648}x^2$$

(bi) $x = 18$ should not be used because the expansion is only valid for $|x/9| < 1 \implies |x| < 9$.

(bii) $x = -2/3$ because it is the value closest to zero, which makes the higher-order terms in the expansion smaller.

(Total for question 10 is 6 marks)

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11 (a)

$$(1+6x)^{\frac{1}{2}} = 1+(\frac{1}{2})(6x)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2}(6x)^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}(6x)^3$$

$$1+3x-\frac{9}{2}x^2+\frac{27}{2}x^3$$

(b) Substitute $x = \frac{1}{12}$ into the expansion to get an estimate for $\sqrt{1+6(1/12)} = \sqrt{1.5} = \sqrt{3/2} = \frac{\sqrt{6}}{2}$. Then multiply the result by 2 to find $\sqrt{6}$.

(Total for question 11 is 5 marks)

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12 (a)

$$(4+x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}(1+\frac{1}{4}x)^{-\frac{1}{2}} = \frac{1}{2}(1+(-\frac{1}{2})(\frac{1}{4}x)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\frac{1}{4}x)^2)$$

$$\frac{1}{2}(1-\frac{1}{8}x+\frac{3}{128}x^2) = \frac{1}{2}-\frac{1}{16}x+\frac{3}{256}x^2$$

(b) Replace x with $-x^2$ in the expansion from part (a):

$$\frac{1}{2}-\frac{1}{16}(-x^2)+\frac{3}{256}(-x^2)^2 = \frac{1}{2}+\frac{1}{16}x^2+\frac{3}{256}x^4$$

(Total for question 12 is 5 marks)

13

(a)
$$9^{-\frac{1}{2}}(1+\frac{2}{9}x)^{-\frac{1}{2}} = \frac{1}{3}(1+(-\frac{1}{2})(\frac{2}{9}x)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\frac{2}{9}x)^2)$$
$$= \frac{1}{3}(1-\frac{1}{9}x+\frac{1}{54}x^2) = \frac{1}{3}-\frac{1}{27}x+\frac{1}{162}x^2$$

(b)
$$(a+bx)(\frac{1}{3}-\frac{1}{27}x+\frac{1}{162}x^2) = 3+x+\ldots$$

Constant term: $\frac{1}{3}a = 3 \implies a = 9$ x term: $\frac{1}{3}b - \frac{1}{27}a = 1$ $\frac{1}{3}b - \frac{1}{27}(9) = 1 \implies \frac{1}{3}b - \frac{1}{3} = 1 \implies \frac{1}{3}b = \frac{4}{3} \implies b = 4$

(Total for question 13 is 7 marks)

14

(a) General term: $\binom{7}{r}(2)^{7-r}(3x)^r$ For x^5, $r=5$: $\binom{7}{5}(2)^2(3x)^5 = 21\times 4\times 243x^5 = 20412x^5$ Coefficient is 20412.

(bi)
$$(1-2x)^{\frac{1}{2}} = 1+(\frac{1}{2})(-2x)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-2x)^2+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{6}(-2x)^3$$
$$1-x-\frac{1}{2}x^2-\frac{1}{2}x^3$$

(bii) $|2x|<1 \implies |x|<\frac{1}{2}$

(biii)
$$\sqrt{98} = \sqrt{100\times 0.98} = 10\sqrt{1-2(0.01)}$$

Let $x=0.01$. Expansion: $1-0.01-0.5(0.01)^2-0.5(0.01)^3 = 1-0.01-0.00005-0.0000005 = 0.9899495$ Estimate: $10\times 0.9899495 = 9.89950$ Calculator: $\sqrt{98} = 9.8994949... \approx 9.89949$ The estimate is very close.

(Total for question 14 is 8 marks)

15

(a)
$$1+(-2)(-2x)+\frac{(-2)(-3)}{2}(-2x)^2+\frac{(-2)(-3)(-4)}{6}(-2x)^3$$
$$1+4x+12x^2+32x^3$$

(b) Geometric series with $a=1$ and $r=2x$.
$$S_\infty = \frac{a}{1-r} = \frac{1}{1-2x}$$

(c) The series is the sum of the expansion in (a) and the series in (b):

$(1+4x+12x^2+32x^3+\ldots)+(1+2x+4x^2+8x^3+\ldots) = 2+6x+16x^2+40x^3+\ldots$

Expression: $(1-2x)^{-2}+(1-2x)^{-1} = \frac{1}{(1-2x)^2}+\frac{1}{1-2x} = \frac{1+(1-2x)}{(1-2x)^2} = \frac{2-2x}{(1-2x)^2}$

(Total for question 15 is 6 marks)

16

$$8^{\frac{1}{3}}(1-\frac{1}{2}x^3)^{\frac{1}{3}} = 2(1+(\frac{1}{3})(-\frac{1}{2}x^3)+\frac{(\frac{1}{3})(-\frac{2}{3})}{2}(-\frac{1}{2}x^3)^2)$$

$$2(1-\frac{1}{6}x^3-\frac{1}{36}x^6) = 2-\frac{1}{3}x^3-\frac{1}{18}x^6$$

(Total for question 16 is 3 marks)

17

(a)

$$(1+ax)^{\frac{1}{2}} = 1+\frac{1}{2}ax+\frac{(\frac{1}{2})(-\frac{1}{2})}{2}(ax)^2 = 1+\frac{1}{2}ax-\frac{1}{8}a^2x^2$$

Coefficients equal: $\frac{1}{2}a = -\frac{1}{8}a^2$ $\frac{1}{8}a^2+\frac{1}{2}a = 0 \implies a^2+4a = 0 \implies a(a+4) = 0$
Since $a \neq 0$, $a = -4$.

(bi)

$$|-4x| < 1 \implies |x| < \frac{1}{4}$$

(bii)

$$1+\frac{1}{2}(-4)x-\frac{1}{8}(-4)^2x^2 = 1-2x-2x^2$$

(Total for question 17 is 5 marks)

18

(a)

$$1+(\frac{1}{2})(3x)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2}(3x)^2 = 1+\frac{3}{2}x-\frac{9}{8}x^2$$

(b)

$$|3x| < 1 \implies |x| < \frac{1}{3}$$

(Total for question 18 is 4 marks)
